

# From Quantity to Number

## Volume 1

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*To M, T, A, P & S.*

## *Preface*

This is a first course in mathematics intended in the first instance for home school students and their parents. In fact, this book derives directly from our own home school math discussions the first year (2015–16) of which it synthesizes.

Our experience with a classical discussion-based approach to home school, a curriculum where the teacher and the student discuss together the topics of the course, is that this method is extremely effective in helping students to learn and in offering an interesting and engaging context for studies. This is as true in mathematics as it is in other subjects. And although many families have recognized the soundness of this method in their non-mathematics studies, it seems much less common for math where often families reluctantly resort to standardized commercial curricula. This book is designed to give interested families the option to pursue also in their math studies a similar discussion-based approach that can be so fruitful and enjoyable for the student and for the teacher.

A central idea to this program is the priority of forming thinking habits of mind, as opposed to drilling mechanical skills. We adhere to the principle that learning a subject should be based always in understanding and reasoning important ideas and not in rote application of a rule or a formula. So we develop and discuss concepts insofar as they offer helpful insight into understanding the topics of our studies. We focus also on how key concepts are related so that we can build up a mathematical understanding featuring coherence from which hopefully emerges for the student the tremendous beauty of mathematics and mathematical inquiry. We also emphasize the multiplicity of approaches that may exist to solve problems and to gain perspective on key ideas.

The main topic in this first set of two volumes is the development of the concept of number. We want to show the concept of number as a beautiful, rich, and evolving idea; an idea that we build up with a progressively increasing mastery of the arithmetic operations. This will lead us from number as a simple tool to help us describe quantity in the most basic sense towards the modern idea of number as the element of a continuum with a fundamen-

tal connection to core concepts underlying more advanced studies such as Calculus.

A subsidiary goal in this text is to introduce as early as possible ideas that are foundational to more advanced mathematics. We believe that these ideas are accessible if they are presented in the right way, that these ideas make discussions more interesting for students and teachers, that they will motivate students to seek out more mathematics, and that they will help future studies by providing a fertile foundation for creative thinking in mathematics. This approach also allows students to begin turning over these ideas in their minds earlier.

By way of reference, our daily math program included two parts. First a discussion on the topics set out in this text. The student participates actively in the flow of the discussion even if sometimes by indicating that a new approach to a topic is required. We kept a blank notebook handy, like a chalk board, during the discussions and we would work through many problems as we thought about the ideas that you can see in the table of contents.

The second essential part of our daily math program was an assignment for the student to work alone. Usually I provided just specifications for the assignments. So for example if you are working on multiplication, then you might indicate 20 multiplication problems say with the numbers 6 and 12 times two digit numbers between 41 and 69. In this way the teacher can tailor assignments based on the student's need while providing the student some leeway to think up problems independently. The student can learn a lot in this.

The discussions set out in this book start from beginning counting and assume no prior mathematical knowledge. Students need not even be reading yet as long as they are ready to follow along in the discussions proposed by their teacher. For the teachers, we recommend reading at least the entire chapter (preferably all the chapters) prior to beginning discussions. A preview of all the chapters will take most teachers about an hour but it will enable the teacher to see how the key ideas and themes develop and also how the examples and exercises might be modeled for additional practice if necessary. This kind of preview will help the teacher to lead discussions effectively throughout the chapters even if the teacher did not pursue math much beyond school. The chapters are not too long but they will provide the teacher with specific guidance. The text is not intended as a verbatim script but rather as a source for specific concepts, explanations, insights, examples, and exercises providing a secure framework for discussions with a clear ordering of ideas and relationships. An important aspect of our approach here is that the teacher has freedom to maneuver (which is more interesting for the teacher and the student) and also to tailor discussions to suit individual student needs.

The first focus in the text is on counting in the usual sense. Chapter one covers beginning counting. Counting up, looking towards the infinite and

counting towards the infinite. We develop the arithmetic operations based on their relatedness and how they provide for counting in new ways. So after a quick overview of numbers in chapter two we turn successively to addition, multiplication, division, subtraction, and fractions. This ordering enables us to build up our work incrementally. Thus, for example, multiplication follows naturally and intuitively from addition—as repeated addition; division flows out of any multiplication expression and so is discussed immediately after multiplication; subtraction extends our understanding of number with the property of direction, and finally fractions synthesize our work on division and counting.

As we become more familiar with counting in this initial approach, we move on to consider another perspective; that of *counting closer*. In this second focus, which we broach in volume 1 and continue in volume 2 (forthcoming), we look carefully at the gaps between numbers and for which we lack a numerical representation until we come up with some ideas on how to express numbers that can describe these missing quantities. In this we use the intuitive geometric representation of the number line depicting numbers as points on the line and showing the *continuum* structure of quantities that we would like to make available in our conception of numbers also. The unbroken line is a simple way to see the goal of a continuous number structure and the difference between that continuum and the discrete collection of numbers in the rational number system.

The text provides examples and exercises. As indicated above, we further recommend assignments comprising *specifications* for problems that the student should devise and solve. The assignment specifications can be modeled on the examples and exercises while also being tailored to each student. In our experience, these student-created exercises offer a good opportunity for students to think about forming questions and provide them with the occasion to consider topics from a different perspective. Students gain insight and also the beneficial practice of active engagement through this effort.

Each student (and teacher) should work patiently with exercises until the concepts are clear and familiar. This work may entail different kinds of effort for different students depending on background. It is also a good idea to keep in mind that working through a math text is more like working through a musical score than like reading a novel. Studying math benefits from careful application and review. It is the work of an iterative process building up a structure of ideas. It is perhaps more similar to a design process than to our usual approach to reading a story book.

Throughout the text, there are several recurring themes. We have sought to identify these themes explicitly because they are interesting in themselves and they help to relate the disparate areas of study. For example, the theme of distinguishing an idea from its representation, the theme of changing perspectives, an introduction to the idea of process and of describing building

numbers or computations as process, and the theme of building and finding structure in numbers. For example, the idea of building whole numbers by incrementation and the idea of building and extending our system of numbers as we discover new structure, patterns, and properties through arithmetic.

We hope that this text will encourage and help other families to include mathematical discussions in their daily study program.

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Tequesta, Florida  
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This book derives from discussions with Thomas and from our work together through many topics.

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